

ON THE DIOPHANTINE PROPERTIES OF  $\lambda$ -EXPANSIONS

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ABSTRACT. For  $\lambda \in (\frac{1}{2}, 1)$  and  $\alpha$ , we consider sets of numbers  $x$  such that for infinitely many  $n$ ,  $x$  is  $2^{-\alpha n}$ -close to some  $\sum_{i=1}^n \omega_i \lambda^i$ , where  $\omega_i \in \{0, 1\}$ . These sets are in Falconer's intersection classes for Hausdorff dimension  $s$  for some  $s$  such that  $-\frac{1}{\alpha} \frac{\log \lambda}{\log 2} \leq s \leq \frac{1}{\alpha}$ . We show that for almost all  $\lambda \in (\frac{1}{2}, \frac{2}{3})$ , the upper bound of  $s$  is optimal, but for a countable infinity of values of  $\lambda$  the lower bound is the best possible result.

## 1. INTRODUCTION

Diophantine approximation deals with the approximation of real numbers by rationals. A classic example is the set  $J(\alpha)$  of all  $\alpha$ -well approximable numbers,

$$J(\alpha) = \{x \in \mathbb{R} : |x - p/q| < q^{-\alpha} \text{ for infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N}\}.$$

Dirichlet showed that  $J(\alpha) = \mathbb{R}$  for  $\alpha = 2$  and Jarník [J] and Besicovitch [B] showed that the Hausdorff dimension of  $J(\alpha)$  is  $2/\alpha$  for all  $\alpha \geq 2$ .

The sets  $J(\alpha)$  belong to a family of sets with an interesting large intersection property, first introduced by Falconer in [F1, F2]. Falconer defined classes  $\mathcal{G}^s$  of  $G_\delta$  subsets of  $\mathbb{R}^n$  with the property that any set in  $\mathcal{G}^s$  has Hausdorff dimension at least  $s$ , and any countable intersection of bi-Lipschitz images of sets from  $\mathcal{G}^s$ , also belongs to  $\mathcal{G}^s$ . There are several equivalent ways to characterise the sets in  $\mathcal{G}^s$  (see [F2]).

Falconer showed that the set  $J(\alpha)$  is in the class  $\mathcal{G}^{2/\alpha}$  [F1]. This implies that any countable intersection of  $J(\alpha)$  with sets from  $\mathcal{G}^{2/\alpha}$  has Hausdorff dimension  $2/\alpha$ .

Real numbers are typically represented by some imperfect truncation of their expansion to some given integer base. This motivates the classification of numbers according to the accuracy of their finite expansions by considering sets of the form,

$$\mathcal{B}(\alpha) = \{x \in \mathbb{R} : |x - p/2^n| < 2^{-\alpha n} \text{ for infinitely many } (p, n) \in \mathbb{Z} \times \mathbb{N}\}.$$

For each  $\alpha$  the set  $\mathcal{B}(\alpha)$  is of Hausdorff dimension  $1/\alpha$ . Moreover, each  $\mathcal{B}(\alpha)$  belongs to the class  $\mathcal{G}^{1/\alpha}$ . We note that  $\mathcal{B}(\alpha) = \mathcal{D}(\alpha) + \mathbb{Z}$  where,

$$\mathcal{D}(\alpha) = \left\{x \in [0, 1] : \left|x - \sum_{i=1}^n \omega_i 2^{-i}\right| < 2^{-n\alpha} \right. \\ \left. \text{for infinitely many } \omega \in \{0, 1\}^n, n \in \mathbb{N} \right\}.$$

For each  $n \in \mathbb{N}$  we let  $\mathcal{D}_n$  denote the set of all  $n$ -th level dyadic sums

$$\mathcal{D}_n := \left\{ \sum_{i=1}^n \omega_i 2^{-i} : \omega \in \{0, 1\}^n \right\}.$$

The fact that  $\mathcal{B}(\alpha)$  belongs to the class  $\mathcal{G}^{1/\alpha}$  is essentially a consequence of the fact that each  $\mathcal{D}_n$  is evenly distributed in  $[0, 1]$ . This motivates the heuristic principle that if  $\{\mathcal{D}_n\}_{n \in \mathbb{N}}$  were replaced by some other family of suitably well distributed sets then we should still obtain a set with large intersection properties.

Now take some  $\lambda \in (\frac{1}{2}, 1)$ . Just as every number between zero and one may be written as a binary expansions, any number  $x \in [0, \lambda(1 - \lambda)^{-1}]$  may be written in the form,

$$x = \sum_{i=1}^{\infty} \omega_i \lambda^i,$$

for some  $(\omega_i)_{i=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ . Following Pollicott and Simon [PS] we refer to  $(\omega_i)_{i=1}^{\infty}$  as the  $\lambda$ -expansion of  $x$ . In this paper we shall study the approximation of real numbers by the finite truncations of their  $\lambda$ -expansions. Hence, we study sets of the form

$$W_{\lambda}(\alpha) = \left\{ x \in [0, \lambda/(1 - \lambda)] : \left| x - \sum_{k=1}^n \omega_k \lambda^k \right| < 2^{-\alpha n}, \right. \\ \left. \text{for infinitely many } \omega \in \{0, 1\}^{\mathbb{N}}, n \in \mathbb{N} \right\}.$$

Since  $W_{\lambda}(\alpha)$  is a subset of  $[0, \lambda/(1 - \lambda)]$  it cannot belong to any class  $\mathcal{G}^s$ . Instead we will consider the corresponding versions of the classes  $\mathcal{G}^s$  for subsets of an interval  $I$ , denoted by  $\mathcal{G}^s(I)$ . It is natural to conjecture that for almost every  $\lambda$ ,  $W_{\lambda}(\alpha)$  belongs to the set  $\mathcal{G}^{1/\alpha}(I)$ . This conjecture is motivated by our heuristic principle combined with results concerning the distribution of the  $n$ -th level  $\lambda$ -sums,

$$\mathcal{D}_{\lambda}(n) := \left\{ \sum_{i=1}^n \omega_i 2^{-i} : \omega \in \{0, 1\}^n \right\}.$$

This topic has attracted a great deal of interest since the time of Erdős [E]. Erdős studied a class of measures known as *infinite Bernoulli convolutions* formed by taking the distributions of the random variable

$$\sum_{k=1}^{\infty} \pm \lambda^k,$$

for some  $\lambda \in (\frac{1}{2}, 1)$ , where in each term  $+$  and  $-$  are chosen independently and with equal probability. Erdős proved the existence of an interval  $(a, 1)$  for which the infinite Bernoulli convolution is absolutely continuous for almost every  $\lambda \in (a, 1)$  [E]. Erdős also proved the existence of a countable family of  $\lambda$  for which the corresponding infinite Bernoulli convolution is not absolutely continuous. Nonetheless it was conjectured that for almost every  $\lambda \in (\frac{1}{2}, 1)$  the corresponding infinite Bernoulli convolution is absolutely continuous. In a breakthrough work of Solomyak this conjecture was answered

in the affirmative [S]. This implies that for typical  $\lambda$  the sums  $\mathcal{D}_\lambda(n)$  are fairly evenly distributed in the sense of Lebesgue.

We shall show that for almost all  $\lambda \in (\frac{1}{2}, \frac{2}{3})$ , the set  $W_\lambda(\alpha)$  belongs to the class  $\mathcal{G}^{\frac{1}{\alpha}}(I)$ . However there is a dense set of  $\lambda$  such that the dimension of  $W_\lambda(\alpha)$  drops below  $1/\alpha$ . We also show that for any  $\lambda \in (\frac{1}{2}, 1)$ , the set  $W_\lambda(\alpha)$  belongs to  $\mathcal{G}^s(I)$ , at least for  $s = -\frac{1}{\alpha} \frac{\log \lambda}{\log 2}$ . We also show that this estimate is sharp in the sense that there exists a countable set of  $\lambda$  for which  $\dim_{\text{H}} W_\lambda(\alpha) = -\frac{1}{\alpha} \frac{\log \lambda}{\log 2}$ , and hence  $W_\lambda(\alpha)$  is not in the class  $\mathcal{G}^s(I)$  for any  $s$  larger than  $-\frac{1}{\alpha} \frac{\log \lambda}{\log 2}$ .

## 2. NOTATION AND STATEMENT OF RESULTS

We begin by defining the classes  $\mathcal{G}^s(I)$  referred to in the introduction. One characterisation of Falconer's class  $\mathcal{G}^s$  is as follows [F2].  $\mathcal{G}^s$  is the set of all  $G_\delta$  sets  $A$  which have the property that for any countable collection  $\{f_j\}_{j \in \mathbb{N}}$  of similarity transformations  $f_j : \mathbb{R} \rightarrow \mathbb{R}$  we have,

$$\dim_{\text{H}} \left( \bigcap_{j \in \mathbb{N}} f_j(A) \right) \geq s.$$

The class  $\mathcal{G}^s(I)$  may be defined in terms of  $\mathcal{G}^s$ .

**Definition 2.1.** Given an interval  $I$ , the class  $\mathcal{G}^s(I)$  is the class of subsets of  $I$  given by  $\mathcal{G}^s(I) := \{A \subseteq I : A + \text{diam}(I) \cdot \mathbb{Z} \in \mathcal{G}^s\}$ .

We let  $I_\lambda$  denote the closed interval  $[0, \lambda/(1 - \lambda)]$  which consists of all points  $x \in \mathbb{R}$  which may be written in the form  $x = \sum_{i=1}^{\infty} \omega_i \lambda^i$  for some  $\omega \in \{0, 1\}^{\mathbb{N}}$ . We shall consider the sets  $W_\lambda(\alpha)$  of points which are  $\alpha$ -well-approximated by  $\lambda$ -expansions,

$$W_\lambda(\alpha) := \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \bigcup_{\omega \in \{0, 1\}^n} \left\{ x \in I_\lambda : \left| x - \sum_{i=1}^n \omega_i \lambda^i \right| < 2^{-n\alpha} \right\}.$$

**Theorem 1.** Choose  $\alpha \in (1, \infty)$ .

- (1) For all  $\lambda \in (\frac{1}{2}, 1)$ ,  $\dim W_\lambda(\alpha) \leq \frac{1}{\alpha}$ ,
- (2) For almost every  $\lambda \in (\frac{1}{2}, \frac{2}{3})$ ,  $W_\lambda(\alpha) \in \mathcal{G}^s(I_\lambda)$  for  $s = \frac{1}{\alpha}$ ,
- (3) For a dense set of  $\lambda \in (\frac{1}{2}, 1)$ ,  $\dim W_\lambda(\alpha) < \frac{1}{\alpha}$ ,
- (4) For all  $\lambda \in (\frac{1}{2}, 1)$ ,  $W_\lambda(\alpha) \in \mathcal{G}^s(I_\lambda)$  for  $s = -\frac{\log \lambda}{\log 2} \frac{1}{\alpha}$ ,
- (5) For a countable set of  $\lambda \in (\frac{1}{2}, 1)$ ,  $\dim W_\lambda(\alpha) = -\frac{\log \lambda}{\log 2} \frac{1}{\alpha}$ .

In addition to Theorem 1 (2) we also have the following upper bound on the dimension of the set of exceptions.

**Theorem 2.** Given  $\alpha > 1$  and  $s \leq \frac{1}{\alpha}$  we have,

$$\dim_{\text{H}} \left\{ \lambda \in \left( \frac{1}{2}, \frac{2}{3} \right) : W_\lambda(\alpha) \notin \mathcal{G}^s(I_\lambda) \right\} \leq s.$$

The remainder of the paper is structured as follows. In Section 3 we prove Theorem 2, which implies Theorem 1 (2). In Section 4 we establish the uniform lower bound given in Theorem 1 (4). In Section 5 we prove the upper bounds in Theorem 1 parts (1), (3) and (5).

## 3. PROOF OF THEOREM 2

In this section we prove Theorem 2. The proof of this theorem is influenced by Rams' work on the dimension of the exceptional set for families of self-similar measures with overlaps [R].

For each  $\lambda \in (\frac{1}{2}, \frac{2}{3})$  and  $k \in \mathbb{N} \cup \{0\}$ ,  $r \in \mathbb{N}$  we define a pair of proximity numbers

$$\begin{aligned} \tilde{P}_n(\lambda, k, r) &:= \#\left\{(\omega, \kappa) \in (\{0, 1\})^n : \left| \sum_{i=1}^n (\omega_i - \kappa_i) \lambda^i \right| \leq r \cdot \lambda^{n+k}\right\}, \\ P_n(\lambda, k, r) &:= \#\left\{(\omega, \kappa) \in (\{0, 1\})^n : \left| \sum_{i=1}^n (\omega_i - \kappa_i) \lambda^i \right| \leq r \cdot \lambda^{n+k} \text{ and } \omega_1 \neq \kappa_1\right\}. \end{aligned}$$

**Lemma 3.1.** *For all  $n \in \mathbb{N}$  and  $k \in \mathbb{N} \cup \{0\}$ ,  $r \in \mathbb{N}$  we have,*

$$\tilde{P}_n(\lambda, k, r) \leq 2^n + \sum_{l=1}^n 2^{n-l} P_l(\lambda, k, r).$$

*Proof.* For notational convenience we let,

$$\mathcal{P}_n(\lambda, k, r) := \left\{(\omega, \kappa) \in (\{0, 1\})^n : \left| \sum_{i=1}^n (\omega_i - \kappa_i) \lambda^i \right| \leq r \cdot \lambda^{n+k}\right\}.$$

We begin by writing,

$$\begin{aligned} (1) \quad \tilde{P}_n(\lambda, k, r) &= \#\{(\omega, \kappa) \in \mathcal{P}_n(\lambda, k, r) : \omega = \kappa\} \\ &\quad + \sum_{l=1}^n \#\{(\omega, \kappa) \in \mathcal{P}_n(\lambda, k, r) : \omega_i = \kappa_i \text{ for } i \leq n-l \\ &\quad \text{and } \omega_{n-l+1} \neq \kappa_{n-l+1}\}. \end{aligned}$$

The cardinality of the first summand is clearly equal to  $2^n$ . Given a pair  $(\omega, \kappa) \in \mathcal{P}_n(\lambda)$  with  $\omega_i = \kappa_i$  for  $i \leq n-l$  and  $\omega_{n-l+1} \neq \kappa_{n-l+1}$ , for some  $l \in \{1, \dots, k\}$  there exists some  $\eta \in \{0, 1\}^{n-l}$  and  $\zeta, \xi \in \{0, 1\}^l$  with  $\eta_1 \neq \zeta_1$  such that  $\omega = \eta\zeta$  and  $\kappa = \eta\xi$ . It follows from the fact that  $(\omega, \kappa) \in \mathcal{P}_n(\lambda, k, r)$  that,

$$\lambda^{n-l} \left| \sum_{i=1}^l (\zeta_i - \xi_i) \lambda^i \right| \leq r \cdot \lambda^{n+k}.$$

Thus,

$$\left| \sum_{i=1}^l (\zeta_i - \xi_i) \lambda^i \right| \leq r \cdot \lambda^{l+k}.$$

It follows that the number of elements of

$$\{(\omega, \kappa) \in \mathcal{P}_n(\lambda, k, r) : \omega_i = \kappa_i \text{ for } i \leq n-l \text{ and } \omega_{n-l+1} \neq \kappa_{n-l+1}\}$$

is equal to  $P_l(\lambda, k, r)$  multiplied by the number of possible initial strings of length  $n - l$ . Thus,

$$\begin{aligned} \# \{(\omega, \kappa) \in \mathcal{P}_n(\lambda, k, r) : \omega_i = \kappa_i \text{ for } i \leq n - l \text{ and } \omega_{n-l+1} \neq \kappa_{n-l+1}\} \\ = 2^{n-l} P_l(\lambda, k, r). \end{aligned}$$

Substituting into equation (1) completes the proof of the Lemma.  $\square$

To prove that  $W_\lambda(\alpha)$  is in  $\mathcal{G}^s(I_\lambda)$  we will need good estimates on the numbers  $P_n(\lambda, k, r)$ . We will get such estimates for almost all  $\lambda \in (\frac{1}{2}, \frac{2}{3})$ , and the first step to get this is using the following lemma.

**Lemma 3.2** (Shmerkin, Solomyak [ShS]). *For any  $\varepsilon > 0$ , there exists some  $\delta > 0$  such that for all polynomials of the form  $g(\lambda) = 1 + \sum_{i=1}^n a_i \lambda^i$  with  $a_i \in \{-1, 0, 1\}$  and for all  $\lambda \in (\frac{1}{2}, \frac{2}{3} - \varepsilon)$  we have  $g'(\lambda) < -\delta$  whenever  $g(\lambda) < \delta$ .*

Given  $n \in \mathbb{N}$ , a pair  $(\omega, \kappa) \in (\{0, 1\}^n)^2$  and  $\gamma > 0$  we let

$$I_n(\omega, \kappa, \gamma) := \left\{ \lambda \in \left( \frac{1}{2}, \frac{2}{3} \right) : \left| \sum_{i=1}^n (\omega_i - \kappa_i) \lambda^i \right| \leq \gamma \right\}.$$

**Lemma 3.3.** *Let  $\delta$  be as in Lemma 3.2. Then for all  $\gamma \in (0, \delta/2)$  and all pairs  $(\omega, \kappa) \in (\{0, 1\}^n)^2$  with  $\omega_1 \neq \kappa_1$ ,  $I_n(\omega, \kappa, \gamma)$  has diameter not exceeding  $4\delta^{-1}\gamma$ .*

*Proof.* Since  $\omega_1 \neq \kappa_1$  we may assume without loss of generality that  $\omega_1 = 1$  and  $\kappa_1 = 0$ . Choose  $\gamma \in (0, \delta/2)$  and all pairs  $(\omega, \kappa) \in (\{0, 1\}^n)^2$  with  $\omega_1 \neq \kappa_1$ . Now let  $g(\lambda) := \sum_{i=1}^n (\omega_i - \kappa_i) \lambda^{i-1}$ , which is of the required form for Lemma 3.2. We note that  $\lambda \in I_n(\omega, \kappa, \gamma)$  implies  $|g(\lambda)| < \gamma/\lambda < \delta$ . By Lemma 3.2  $g'(\lambda) < -\delta$  whenever  $g(\lambda) < \delta$ . Suppose  $I_n(\omega, \kappa, \gamma) \neq \emptyset$  and choose  $\lambda_0 := \inf I_n(\omega, \kappa, \gamma)$ . It follows from Rolle's theorem that for all  $\lambda \geq \lambda_0$ ,  $g(\lambda) \leq \gamma < \delta$  and hence,  $g'(\lambda) < -\delta$ . Hence,  $I_n(\omega, \kappa, \gamma) \subseteq [\lambda_0, \lambda_0 + 4\delta^{-1}\gamma]$ .  $\square$

Using the following result by Rams [R] we will prove our desired estimates for the numbers  $P_n(\lambda, k, r)$ .

**Lemma 3.4** (Rams [R]). *Suppose we have a family of sets  $\{E_i\}_i$  with  $E_i$  of diameter  $d_i$ . Let  $\rho > 0$  be some positive real number and  $b \in \mathbb{N}$ . Then, the set of points which belong to at least  $b$  of the sets  $E_i$  may be covered by some family of intervals  $\{\tilde{E}_j\}_j$  so that  $\tilde{E}_j$  has diameter  $\tilde{d}_j$  with  $\sup_j \tilde{d}_j \leq 4 \sup_i d_i$  and*

$$\sum_j \tilde{d}_j^\rho \leq 4^\rho \cdot \frac{1}{b} \sum_i d_i^\rho.$$

For each  $s$  we shall let

$$A(s) := \bigcup_{r \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \bigcup_{k \geq 0} \left\{ \lambda \in \left( \frac{1}{2}, \frac{2}{3} \right) : P_n(\lambda, k, r) > 4^n \lambda^{s(n+k)} \right\}.$$

**Lemma 3.5.** *For all  $s \in (0, 1)$  we have  $\dim_H A(s) \leq s$ .*

*Proof.* Choose  $\rho > s$  and take some  $r \in \mathbb{N}$ . Take  $n \in \mathbb{N}$  with  $\lambda^n < \delta/2$ . Note that each  $\lambda \in (\frac{1}{2}, \frac{2}{3})$  with  $P_n(\lambda, k, r) > 4^n \lambda^{s(n+k)}$  is contained within  $I_n(\omega, \kappa, r \cdot \lambda^{n+k})$  for at least  $\lceil 4^n \lambda^{s(n+k)} \rceil$  pairs  $(\omega, \kappa) \in (\{0, 1\}^n)^2$ . Now by Lemma 3.3 each  $I_n(\omega, \kappa, r \cdot \lambda^{n+k})$  has diameter not exceeding  $4\delta^{-1}r\lambda^{n+k}$ . Thus, by Lemma 3.4 we may cover

$$\left\{ \lambda \in \left( \frac{1}{2}, \frac{2}{3} \right) : P_n(\lambda, k, r) > 4^n \lambda^{s(n+k)} \right\}$$

with a family of sets  $A_i^n(s, k)$  of diameter no greater than  $16r\delta^{-1}\lambda^{n+k}$  and satisfying,

$$\begin{aligned} \sum_i \text{diam}(A_i^n(s, k))^\rho &\leq 4^\rho \cdot (4^{-n} \lambda^{-s(n+k)}) \cdot \\ &\quad \cdot \sum_{(\omega, \kappa) \in (\{0, 1\}^n)^2} \text{diam}(I_n(\omega, \kappa, r\lambda^{n+k}))^\rho \\ &\leq (4r)^\rho \cdot (4^{-n} \lambda^{-s(n+k)}) \cdot 2^{2n} \cdot (4\delta^{-1} \lambda^{n+k})^\rho \\ &\leq (16r/\delta)^\rho \lambda^{(n+k)(\rho-s)}. \end{aligned}$$

Consequently, we may cover

$$\bigcup_{k \geq 0} \left\{ \lambda \in \left( \frac{1}{2}, \frac{2}{3} \right) : P_n(\lambda, k, r) > 4^n \lambda^{s(n+k)} \right\}$$

with sets  $A_i^n(s, k)$  of diameter no greater than  $16r\delta^{-1}\lambda^n$  and satisfying,

$$\sum_k \sum_i \text{diam}(A_i^n(s, k))^\rho \leq (16r/\delta)^\rho (1 - \lambda^{\rho-s})^{-1} \lambda^{n(\rho-s)}.$$

It follows that for each  $m \in \mathbb{N}$ ,

$$\bigcup_{n \geq m} \bigcup_{k \geq 0} \left\{ \lambda \in \left( \frac{1}{2}, \frac{2}{3} \right) : P_n(\lambda, k, r) > 4^n \lambda^{s(n+k)} \right\}$$

may be covered by a family of sets  $\bigcup_{n \geq m} \bigcup_k \bigcup_i A_i^n(s, k)$  of diameter not exceeding  $16r\delta^{-1}\lambda^m$  with

$$\begin{aligned} \sum_{n \geq m} \sum_k \sum_i \text{diam}(A_i^n(s, k))^\rho &\leq (16r/\delta)^\rho (1 - \lambda^{\rho-s})^{-1} \cdot \sum_{n \geq m} \lambda^{n(\rho-s)} \\ &\leq (16r/\delta)^\rho (1 - \lambda^{\rho-s})^{-2} \lambda^{m(\rho-s)}. \end{aligned}$$

For every  $m \in \mathbb{N}$  we have,

$$\begin{aligned} &\bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \bigcup_{k \geq 0} \left\{ \lambda \in \left( \frac{1}{2}, \frac{2}{3} \right) : P_n(\lambda, k, r) > 4^n \lambda^{s(n+k)} \right\} \\ &\subseteq \bigcup_{n \geq m} \bigcup_{k \geq 0} \left\{ \lambda \in \left( \frac{1}{2}, \frac{2}{3} \right) : P_n(\lambda, k, r) > 4^n \lambda^{s(n+k)} \right\}. \end{aligned}$$

Thus,

$$\dim_H \left( \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \bigcup_{k \geq 0} \left\{ \lambda \in \left( \frac{1}{2}, \frac{2}{3} \right) : P_n(\lambda, k, r) > 4^n \lambda^{s(n+k)} \right\} \right) \leq \rho.$$

$A(s)$  is a countable union of such sets and so  $\dim_{\mathbb{H}} A(s) \leq \rho$ . Since  $\rho > s$  was arbitrary the Lemma holds.  $\square$

Let  $\mathcal{D} = \{0, 1\}$ . For a natural number  $n$  we denote by  $\mathcal{D}^n$  the set of words  $(\omega_1, \omega_2, \dots, \omega_n)$  of length  $n$  such that each  $\omega_k$  is in  $\mathcal{D}$ . Similarly we denote the set of all such infinite sequences by  $\Sigma$ . If  $\omega$  is an element of  $\Sigma$  or  $\mathcal{D}^l$  with  $l \geq n$ , then we let  $\omega|n$  denote the element in  $\mathcal{D}^n$  such that  $\omega$  and  $\omega|n$  are equal on the first  $n$  places. Given an  $\omega \in \mathcal{D}^n$  we define a function  $g_\omega(x) = \sum_{i=1}^n \omega_i \lambda^i + \lambda^n x$ .

**Lemma 3.6.** *Given a similarity map  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f: x \mapsto rx + t$  for some fixed  $r, t \in \mathbb{R}$ , together with any closed interval  $A$  with non-empty interior and  $\text{diam}(A) < r \text{diam}(I_\lambda)$  there exists an integer  $n(A, f) \in \mathbb{Z}$  and a finite string  $\omega = \omega(A, f) \in \mathcal{D}^\theta$ , with length  $\theta$  depending only on the magnitude of the derivative  $|f'|$  and the diameter  $\text{diam}(A)$  of  $A$ , such that the interval  $f(g_\omega(I_\lambda) + n(A, f) \cdot \text{diam}(I_\lambda))$  is contained within  $A$  and has diameter at least  $\lambda/4 \cdot \text{diam}(A)$ .*

*Proof.* Since  $\text{diam}(A) < r \text{diam}(I_\lambda)$ ,  $\text{diam}(f^{-1}(A)) < \text{diam}(I_\lambda)$ . Hence, the closed interval  $f^{-1}(A)$  intersects at most two of the intervals

$$\{I_\lambda + n \text{diam}(I_\lambda)\}_{n \in \mathbb{Z}}.$$

As such, we may choose  $n(A, f) \in \mathbb{Z}$  so that,

$$\text{diam}(f^{-1}(A) \cap (I_\lambda + n(A, f) \text{diam}(I_\lambda))) \geq \frac{1}{2} \cdot \text{diam}(f^{-1}(A)).$$

Equivalently,  $\text{diam}(Z) \geq \frac{1}{2} \cdot \text{diam}(f^{-1}(A))$  where

$$Z = (f^{-1}(A) - n(A, f) \text{diam}(I_\lambda)) \cap I_\lambda.$$

Let  $x$  denote the midpoint of  $Z$ . Since  $x \in I_\lambda$  we may write  $x = \sum_{i=1}^\infty \omega_i \lambda^i = \bigcap_{n \in \mathbb{N}} g_{\omega|n}(I_\lambda)$ . We choose  $\theta$  so that

$$\theta := \left\lfloor \frac{\log((1 - \lambda) \text{diam}(A)/4|f'|)}{\log \lambda} \right\rfloor.$$

In particular,  $\theta$  depends only upon the magnitude of the derivative  $|f'|$  and the diameter  $\text{diam}(A)$  of  $A$ . Since  $f$  is a similarity and  $I_\lambda$  is of diameter  $\lambda/(1 - \lambda)$ , it follows that

$$\begin{aligned} \text{diam}(g_{\omega|_\theta}(I_\lambda)) &= \frac{\lambda^{\theta+1}}{1 - \lambda} \\ &< \frac{\text{diam}(A)}{4r} \\ &= \frac{1}{2} \cdot \frac{\text{diam}(f^{-1}(A))}{2} \\ &< \frac{1}{2} \cdot \text{diam}(Z). \end{aligned}$$

Since  $x$  is the midpoint of  $Z$  and  $g_{\omega|_\theta}(I_\lambda)$  contains  $x$  we have

$$g_{\omega|_\theta}(I_\lambda) \subseteq Z \subseteq f^{-1}(A) - n(A, f) \cdot \text{diam}(I_\lambda).$$

Hence,

$$f(g_{\omega|_\theta}(I_\lambda) + n(A, f) \cdot \text{diam}(I_\lambda)) \subseteq A.$$

Moreover,

$$\begin{aligned} \text{diam}(g_{\omega|\theta}(I_\lambda)) &= \frac{\lambda^{\theta+1}}{1-\lambda} \\ &\geq \lambda \cdot \frac{\text{diam}(A)}{4r}. \end{aligned}$$

Thus,

$$\text{diam}(f(g_{\omega|\theta}(I_\lambda) + n(A, f) \cdot \text{diam}(I_\lambda))) \geq \frac{\lambda}{4} \cdot \text{diam}(A). \quad \square$$

Given a positive number  $r > 0$  and a finite set  $\Omega$  and two functions  $\varphi_1, \varphi_2 : \Omega \rightarrow \mathbb{R}$  we shall let

$$N_r(\varphi_1, \varphi_2) := \# \{(x, y) \in \Omega^2 : |\varphi_1(x) - \varphi_2(y)| \leq r\}.$$

**Lemma 3.7.** *Given  $r > 0$ , any finite set  $\Omega$ , any function  $\varphi : \Omega \rightarrow \mathbb{R}$  and any  $t \in \mathbb{R}$ , we have  $N_r(\varphi, \varphi + t) \leq 4 \cdot N_r(\varphi, \varphi)$ .*

*Proof of Lemma 3.7.* Since the inequality  $|\varphi_1(x) - \varphi_2(y)| \leq r$  holds if and only if  $|\varphi_1(x)/r - \varphi_2(y)/r| \leq 1$ , it is sufficient to prove the lemma in the case  $r = 1$ .

For each  $n \in \mathbb{Z}$  we let  $a_n := \#(\Omega \cap \varphi^{-1}[n, n+1))$ . Given any pair  $(a, b) \in \Omega^2$  with  $\varphi(a), \varphi(b) \in [n, n+1)$  for some  $n \in \mathbb{Z}$  we have  $|\varphi(a) - \varphi(b)| \leq 1$ . For each  $n \in \mathbb{Z}$  there are at least  $a_n^2$  such pairs, so  $N_1(\varphi, \varphi) \geq \sum_{n \in \mathbb{Z}} a_n^2$ .

Now suppose  $a, b \in \Omega$ ,  $\varphi(a) \in [n, n+1)$ ,  $|\varphi(a) - (\varphi(b) + t)| \leq 1$ . Since  $n \leq \varphi(a) < n+1$ , so  $n-1 \leq \varphi(b) + t < n+2$ , and so

$$n - (\lceil t \rceil + 1) \leq n - 1 - t \leq \varphi(b) < n + 2 - t < (n - (\lfloor t \rfloor - 1)) + 1.$$

Hence,  $\varphi(b)$  is in  $[n-p, n-p+1)$  for some integer  $p$  with  $\lfloor t \rfloor - 1 \leq p \leq \lceil t \rceil + 1$ . Thus, for each  $a \in \Omega$  with  $\varphi(a) \in [n, n+1)$  we have

$$\begin{aligned} &\#\{b \in \Omega : |\varphi(a) - (\varphi(b) + t)| \leq 1\} \\ &\leq \sum_{\lfloor t \rfloor - 1 \leq p \leq \lceil t \rceil + 1} \#(\Omega \cap \varphi^{-1}[n-p, n-p+1)) \\ &= \sum_{\lfloor t \rfloor - 1 \leq p \leq \lceil t \rceil + 1} a_{n-p}. \end{aligned}$$

Thus, for each  $n \in \mathbb{N}$ ,

$$\begin{aligned} &\#\{(a, b) \in \Omega^2 : \varphi(a) \in [n, n+1), |\varphi(a) - (\varphi(b) + t)| \leq 1\} \\ &\leq \sum_{\lfloor t \rfloor - 1 \leq p \leq \lceil t \rceil + 1} a_n \cdot a_{n-p}. \end{aligned}$$



Hence,  $N_1(\varphi, \varphi + t) \leq \sum_{n \in \mathbb{Z}} \sum_{[t]-1 \leq p \leq [t]+1} a_n a_{n-p}$ . Thus, by Cauchy-Schwarz we have,

$$\begin{aligned} N_1(\varphi, \varphi + t) &\leq \sum_{[t]-1 \leq p \leq [t]+1} \sum_{n \in \mathbb{Z}} a_n a_{n-p} \\ &\leq \sum_{[t]-1 \leq p \leq [t]+1} \sqrt{\sum_{n \in \mathbb{Z}} a_n^2 \cdot \sum_{n \in \mathbb{Z}} a_{n-p}^2} \\ &\leq \sum_{[t]-1 \leq p \leq [t]+1} \sum_{n \in \mathbb{N}} a_n^2 \\ &\leq 4N_1(\varphi, \varphi). \end{aligned} \quad \square$$

**Remark 3.1.** *It is natural to ask whether or not 4 is the optimal constant possible in Lemma 3.7. Matthew Aldridge has provided an inductive demonstration that  $N_r(\varphi, \varphi + t) < 2 \cdot N_r(\varphi, \varphi)$ , whilst Oliver Roche-Newton has produced a family of counterexamples showing that such a bound is optimal.*

**Lemma 3.8.** *Suppose  $\lambda \notin A(s)$  and  $r \in \mathbb{N}$ . Then there exists a constant  $C(r) > 0$ , such that for all  $n \in \mathbb{N}$  and all  $k \in \mathbb{N} \cup \{0\}$ ,*

$$\tilde{P}_n(\lambda, k, r) \leq C(r) \cdot 2^n + 4^n n \lambda^{s(n+k)}.$$

*Proof.* Suppose  $\lambda \notin A(s)$  and  $r \in \mathbb{N}$ . Then there exists some  $N_0 \in \mathbb{N}$  such that for all  $n \geq N_0$  and all  $k \in \mathbb{N} \cup \{0\}$ ,  $P_n(\lambda, k, r) \leq 4^n \lambda^{s(n+k)}$ . Thus, if we take  $C := 1 + \sum_{l=1}^{N_0} 2^{-l} P_l(\lambda, 0, r)$  then by Lemma 3.1 then we have,

$$\begin{aligned} \tilde{P}_n(\lambda, k, r) &\leq 2^n + \sum_{l=1}^n 2^{n-l} P_l(\lambda, k, r) \\ &\leq 2^n \left( 1 + \sum_{l=1}^{N_0} 2^{-l} P_l(\lambda, k, r) \right) + \lambda^{sk} \cdot \sum_{l=N_0+1}^n 2^{n-l} \cdot (4\lambda^s)^l \\ &\leq 2^n \left( 1 + \sum_{l=1}^{N_0} 2^{-l} P_l(\lambda, 0, r) \right) + \lambda^{sk} \cdot \sum_{l=N_0+1}^n (4\lambda^s)^n \\ &\leq C \cdot 2^n + 4^n n \lambda^{s(n+k)}, \end{aligned}$$

where we used the fact that  $\lambda \geq \frac{1}{2}$ , so  $4\lambda^s \geq 2$ .  $\square$

**Proposition 3.1.** *Suppose  $\lambda \notin A(s)$  for some  $s \leq \frac{1}{\alpha}$ . Then  $W_\lambda(\alpha) \in \mathcal{G}^s(I_\lambda)$ .*

*Proof.* To prove the proposition we begin by fixing  $\lambda \notin A(s)$ ,  $\alpha > 1$  and a sequence of similarity maps  $\{f_j\}_{j \in \mathbb{N}}$ . We shall show that

$$\dim_{\text{H}} \left( \bigcap_{j \in \mathbb{N}} f_j(W_\lambda(\alpha) + \text{diam}(I_\lambda) \cdot \mathbb{Z}) \right) \geq s.$$

To do so we shall construct a subset  $\Lambda \subset \bigcap_{j \in \mathbb{N}} f_j(W_\lambda(\alpha) + \text{diam}(I_\lambda) \cdot \mathbb{Z})$  supporting a measure  $\nu$  with correlation dimension  $s$ . Without loss of generality we may assume that  $f_1 : x \mapsto 2x$ . We begin by choosing a sequence of natural numbers  $(j(q))_{q \in \mathbb{N} \cup \{0\}}$  so that  $j(0) = 1$  and for each  $k \in \mathbb{N}$ ,

$$(2) \quad \#\{q : j(q) = k\} = \infty.$$

Let  $\Sigma_* = \{\emptyset\} \cup_n \mathcal{D}^n$ . We shall recursively construct sequences of integers  $(\gamma_q)_{q \in \mathbb{N}}$ ,  $(\hat{\gamma}_q)_{q \in \mathbb{N}}$ ,  $(\theta_q)_{q \in \mathbb{N}}$  and  $(m_q)_{q \in \mathbb{N}}$  along with closed intervals  $(\Delta_\omega)_{\omega \in \Sigma_*}$  and  $(\hat{\Delta}_\omega)_{\omega \in \Sigma_*}$  and positive reals  $(\delta_n)_{n \in \mathbb{N} \cup \{0\}}$ ,  $(\hat{\delta}_n)_{n \in \mathbb{N} \cup \{0\}}$  with the property that for any  $\omega \in \Sigma_*$  and  $\eta \in \mathcal{D}$ ,

$$\Delta_\omega \supseteq \hat{\Delta}_\omega \supseteq \Delta_{\omega\eta}.$$

Moreover, given any word  $\omega \in \mathcal{D}^n$  for some  $n \in \mathbb{N} \cup \{0\}$  we have  $\text{diam}(\Delta_\omega) = \delta_n$  and  $\text{diam}(\hat{\Delta}_\omega) = \hat{\delta}_n$ . We also have  $\hat{\delta}_n \leq \delta_n \leq \lambda^{n+1}/(1-\lambda)$  for all  $n \in \mathbb{N} \cup \{0\}$ . In addition,  $\lambda^{\gamma_q} < \|f'_{j(q)}\|_\infty$  for  $q \geq 1$ .

First let  $\gamma_0 = \hat{\gamma}_0 = \theta_0 = m_0 = 0$ ,  $\Delta_\emptyset = \hat{\Delta}_\emptyset = I_\lambda$  and  $\delta_0 = \hat{\delta}_0 = \lambda/(1-\lambda)$ .

Suppose we have chosen  $\gamma_l$ ,  $\theta_l$  and  $m_l$  for  $l \leq q$  and for all  $n \leq \Gamma(q) := \sum_{l \leq q} \gamma_l$  we have defined  $\delta_n$ ,  $\hat{\delta}_n$  and for  $\omega \in \mathcal{D}^n$  we have  $\Delta_\omega$  and  $\hat{\Delta}_\omega$ , all satisfying the required properties.

For the inductive step we first apply Lemma 3.6 to obtain  $(\omega(\kappa))_{\kappa \in \mathcal{D}^{\Gamma(q)}}$  and  $(n(\kappa))_{\kappa \in \mathcal{D}^{\Gamma(q)}}$  with  $n(\kappa) = n(\hat{\Delta}_\kappa, f_{j(q)}) \in \mathbb{Z}$  and  $\omega(\kappa) = \omega(\hat{\Delta}_\kappa, f_{j(q)}) \in \Sigma_*$  for each  $\kappa \in \mathcal{D}^{\Gamma(q)}$  so that,

- (1)  $f_{j(q)}(g_{\omega(\kappa)}(I_\lambda) + n(\kappa) \text{diam}(I_\lambda)) \subseteq \hat{\Delta}_\kappa$ ,
- (2)  $\text{diam}(f_{j(q)}(g_{\omega(\kappa)}(I_\lambda) + n(\kappa) \text{diam}(I_\lambda))) \geq \frac{\lambda}{4} \cdot \text{diam}(\hat{\Delta}_\kappa)$ .

By supposition,  $\text{diam}(\hat{\Delta}_\kappa) = \delta_{\Gamma(q)}$  for all  $\kappa \in \mathcal{D}^{\Gamma(q)}$ . Consequently, by Lemma 3.6 the length of  $|\omega(\kappa)|$  is uniform over all  $\kappa \in \mathcal{D}^{\Gamma(q)}$ . We denote this uniform length by  $\theta_{q+1}$ .

Choose  $\gamma_{q+1}, \hat{\gamma}_{q+1} \in \mathbb{N}$  so that,

$$\begin{aligned} \gamma_{q+1} &> q\gamma_q\theta_{q+1} \cdot (-\log \delta_{\Gamma(q)}), \\ \gamma_{q+1} &> \frac{\log |f'_{j(q+1)}|}{\log \lambda}, \\ \hat{\gamma}_{q+1} &= \gamma_{q+1} + \theta_{q+1}. \end{aligned}$$

and let

$$m_{q+1} := \left\lfloor \left( \frac{\log 2^{-\alpha}}{\log \lambda} - 1 \right) \hat{\gamma}_{q+1} - \frac{\log(1-\lambda)}{\log \lambda} \right\rfloor + 1,$$

so that

$$(3) \quad \lambda^{\hat{\gamma}_{q+1} + m_{q+1}} / (1-\lambda) < 2^{-\alpha \hat{\gamma}_{q+1}} \leq \lambda^{\hat{\gamma}_{q+1} + m_{q+1} - 1} / (1-\lambda).$$

Given  $\kappa \in \mathcal{D}^{\Gamma(q)}$  and  $\tau \in \mathcal{D}^l$  for some  $l \leq \gamma_{q+1}$  we define

$$\Delta_{\kappa\tau} := f_{j(q)}(g_{\omega(\kappa)} \circ g_\tau(I_\lambda) + n(\kappa) \cdot \text{diam}(I_\lambda)).$$

Thus, for all  $\omega \in \mathcal{D}^{\Gamma(q)+l}$  for some  $l \leq \gamma_{q+1}$  we have,

$$\text{diam}(\Delta_\omega) = \delta_{\Gamma(q)+l} := |f'_{j(q)}| \cdot \lambda^{\theta_{q+1}+l+1} / (1-\lambda).$$

Moreover, for  $l < \gamma_{q+1}$  we let  $\hat{\Delta}_{\kappa\tau} := \Delta_{\kappa\tau}$  and for  $l = \gamma_{q+1}$ ,

$$\hat{\Delta}_{\kappa\tau} := f_{j(q)}(g_{\omega(\kappa)} \circ g_\tau \circ (g_0)^{m_{q+1}}(I_\lambda) + n(\kappa) \cdot \text{diam}(I_\lambda)).$$

Hence, for all  $\omega \in \mathcal{D}^{\Gamma(q)+l}$  for some  $l < \gamma_{q+1}$  we have,

$$\text{diam}(\hat{\Delta}_\omega) = \hat{\delta}_{\Gamma(q)+l} := \delta_{\Gamma(q)+l},$$

and for  $\omega \in \mathcal{D}^{\Gamma(q+1)}$

$$\text{diam}(\hat{\Delta}_\omega) = \hat{\delta}_{\Gamma(q+1)} := |f'_{j(q)}| \cdot \lambda^{\theta_{q+1} + \gamma_{q+1} + m_{q+1} + 1} / (1 - \lambda).$$

It follows that for all  $\eta \in \mathcal{D}^{\Gamma(q+1)}$

$$(4) \quad \hat{\Delta}_\eta \subseteq f_{j(q)} \left( \bigcup_{\omega \in \mathcal{D}^{\hat{\gamma}_{q+1}}} \left\{ x \in I_\lambda : \left| x - \sum_{i=1}^{\hat{\gamma}_{q+1}} \omega_i \lambda^i \right| < 2^{-\hat{\gamma}_{q+1}\alpha} \right\} + \mathbb{Z} \text{diam}(I_\lambda) \right).$$

In this way we have defined two families of closed intervals  $(\Delta_\omega)_{\omega \in \Sigma_*}$  and  $(\hat{\Delta}_\omega)_{\omega \in \Sigma_*}$  with the property that for any  $\omega \in \Sigma_*$  and  $\eta \in \mathcal{D}$ ,

$$\Delta_\omega \supseteq \hat{\Delta}_\omega \supseteq \Delta_{\omega\eta},$$

and given any  $\omega \in \mathcal{D}^n$ ,  $\text{diam}(\Delta_\omega) \leq \lambda^{n+1} / (1 - \lambda)$ . Thus, we may define a map  $\pi : \Sigma \rightarrow I_\lambda$  by

$$\pi(\omega) := \bigcap_{n \in \mathbb{N}} \Delta_{\omega|n} = \bigcap_{n \in \mathbb{N}} \hat{\Delta}_{\omega|n}.$$

By construction we also have  $\delta_n \geq \lambda^2 / 2\hat{\delta}_{n-1}$  for all  $n \in \mathbb{N}$ .

We let  $\Lambda := \pi(\Sigma)$ . By Equations (4) and (2) we have

$$\Lambda \subset \bigcap_{j \in \mathbb{N}} f_j(W_\lambda(\alpha) + \mathbb{Z} \cdot \text{diam}(I_\lambda)).$$

Thus, to complete the proof it suffices to show that  $\dim_{\text{H}} \Lambda \geq s$ . In order to do this we shall define a measure supported on  $\Lambda$  with the property

$$\dim_{\text{C}}(\nu) := \liminf_{r \rightarrow 0} \frac{1}{\log r} \log \int \nu(B_r(x)) d\nu(x) \geq s.$$

That is, the correlation dimension  $\dim_{\text{C}}(\nu)$  of  $\nu$  is at least  $s$ . This implies that the Hausdorff dimension of  $\nu$  and hence  $X$  is at least  $s$  (see [PD, Section 17]).

We do this by taking  $\mu$  to be the  $(\frac{1}{2}, \frac{1}{2})$ -Bernoulli measure on  $\Sigma$  and  $\nu$  its projection by  $\pi$ ,  $\nu := \mu \circ \pi^{-1}$ .

In order to estimate  $\dim_{\text{C}}(\nu)$  we require good upper bounds on the number of intervals  $\hat{\Delta}_\omega$  of a given level which are close to one another.

**Lemma 3.9.** *Suppose  $\rho > 1$  and  $\lambda \notin A(s)$ . Then there exists a constant  $C$  depending only on  $\rho$  and  $\lambda$  such that for any pair  $\eta, \zeta \in \mathcal{D}^{\Gamma(q)}$  for some  $q \in \mathbb{N}$ ,  $n = l + \Gamma(q)$  for some  $l \leq \gamma_{q+1}$  and  $\hat{\delta}_n \leq t \leq \rho \cdot \delta_n$  we have,*

$$\# \left\{ (\kappa, \tau) \in \mathcal{D}^l : d(\hat{\Delta}_{\eta\kappa}, \hat{\Delta}_{\zeta\tau}) < t \right\} \leq C \cdot 2^l + 4^{l\lambda} \cdot (\rho\lambda)^{-s} \left( \frac{t}{\hat{\delta}_{n-l}} \right)^s.$$

*Proof.* We begin by noting that for each pair  $(\kappa, \tau) \in \mathcal{D}^l$  we have

$$\begin{aligned} f_{j(q)} \circ g_{\omega(\eta)} \circ g_\kappa(0) + f_{j(q)}(n(\eta)) &\in \hat{\Delta}_{\eta\kappa} \\ f_{j(q)} \circ g_{\omega(\zeta)} \circ g_\tau(0) + f_{j(q)}(n(\zeta)) &\in \hat{\Delta}_{\zeta\tau}. \end{aligned}$$

Since every  $\hat{\Delta}_{\eta\kappa}, \hat{\Delta}_{\zeta\tau}$  has diameter  $\hat{\delta}_n$ ,  $t \geq \hat{\delta}_n$  we have,

$$\begin{aligned} & \# \{ (\kappa, \tau) \in \mathcal{D}^l : d(\hat{\Delta}_{\eta\kappa}, \hat{\Delta}_{\zeta\tau}) < t \} \\ & \leq \# \left\{ (\kappa, \tau) \in \mathcal{D}^l : \left| f_{j(q)} \circ g_{\omega(\eta)} \circ g_{\kappa}(0) - f_{j(q)} \circ g_{\omega(\zeta)} \circ g_{\tau}(0) + \right. \right. \\ & \quad \left. \left. + (f_{j(q)}(n(\eta)) - f_{j(q)}(n(\zeta))) \right| < 2t \right\} \end{aligned}$$

Since  $f_{j(q)}$  is affine we have,

$$\begin{aligned} & f_{j(q)} \circ g_{\omega(\zeta)} \circ g_{\tau}(0) \\ & = (f_{j(q)} \circ g_{\omega(\eta)} \circ g_{\tau}(0) + (f_{j(q)} \circ g_{\omega(\zeta)}(0) - f_{j(q)} \circ g_{\omega(\eta)}(0))). \end{aligned}$$

By applying Lemma 3.7 we obtain

$$\begin{aligned} & \# \left\{ (\kappa, \tau) \in \mathcal{D}^l : \left| f_{j(q)} \circ g_{\omega(\eta)} \circ g_{\kappa}(0) - f_{j(q)} \circ g_{\omega(\zeta)} \circ g_{\tau}(0) + \right. \right. \\ & \quad \left. \left. + (f_{j(q)}(n(\eta)) - f_{j(q)}(n(\zeta))) \right| < 2t \right\} \\ & \leq 4\# \left\{ (\kappa, \tau) \in \mathcal{D}^l : \left| f_{j(q)} \circ g_{\omega(\eta)} \circ g_{\kappa}(0) - f_{j(q)} \circ g_{\omega(\eta)} \circ g_{\tau}(0) \right| < 2t \right\}. \end{aligned}$$

We note that

$$\begin{aligned} \|(f_{j(q)} \circ g_{\omega(\eta)})'\|_{\infty} & \geq \frac{1-\lambda}{\lambda} \cdot \text{diam}(f_{j(q)} \circ g_{\omega(\kappa)}(I_{\lambda})) \\ & \geq \frac{1-\lambda}{\lambda} \cdot \frac{\lambda}{2} \cdot \text{diam}(\hat{\Delta}_{\kappa}) \\ & = (1-\lambda) \cdot \hat{\delta}_{n-l}/2. \end{aligned}$$

Piecing the above together we have

$$\begin{aligned} & \# \{ (\kappa, \tau) \in \mathcal{D}^l : d(\hat{\Delta}_{\eta\kappa}, \hat{\Delta}_{\zeta\tau}) < t \} \\ & \leq 4 \cdot \left\{ (\kappa, \tau) \in \mathcal{D}^l : \left| f_{j(q)} \circ g_{\omega(\eta)} \circ g_{\kappa}(0) - \right. \right. \\ & \quad \left. \left. - f_{j(q)} \circ g_{\omega(\eta)} \circ g_{\tau}(0) \right| < 2t \right\} \\ & \leq 4 \cdot \# \{ (\kappa, \tau) \in \mathcal{D}^l : |g_{\kappa}(0) - g_{\tau}(0)| < 2t \|(f_{j(q)} \circ g_{\omega(\eta)})'\|_{\infty}^{-1} \} \\ & \leq 4 \cdot \# \left\{ (\kappa, \tau) \in \mathcal{D}^l : |g_{\kappa}(0) - g_{\tau}(0)| < \frac{4t}{(1-\lambda)\hat{\delta}_{n-l}} \right\}. \end{aligned}$$

Since  $\delta_n \leq \hat{\delta}_{n-l} \cdot \lambda^l$  and  $t \leq \rho\delta_n$  we have,

$$\frac{4t}{(1-\lambda)\hat{\delta}_{n-l}} \leq \frac{4\rho}{1-\lambda} \cdot \lambda^l.$$

Now choose  $k \in \mathbb{N} \cup \{0\}$  so that

$$\frac{4\rho}{1-\lambda} \cdot \lambda^{l+k+1} < \frac{4t}{(1-\lambda)\hat{\delta}_{n-l}} \leq \frac{4\rho}{1-\lambda} \cdot \lambda^{l+k}.$$

By applying Lemma 3.8 we have

$$\begin{aligned}
& \#\{(\kappa, \tau) \in \mathcal{D}^l : d(\hat{\Delta}_{\eta\kappa}, \hat{\Delta}_{\zeta\tau}) < t\} \\
& \leq 4 \cdot \#\{(\kappa, \tau) \in \mathcal{D}^l : |g_\kappa(0) - g_\tau(0)| < 4\rho(1-\lambda)^{-1}\lambda^{l+k}\} \\
& = 4 \cdot \tilde{P}_l\left(\lambda, k, \frac{4\rho}{1-\lambda}\right) \\
& \leq C\left(\frac{4\rho}{1-\lambda}\right) \cdot 2^l + 4^l l \lambda^{s(l+k)} \\
& \leq C\left(\frac{4\rho}{1-\lambda}\right) \cdot 2^l + 4^l l \cdot (\rho\lambda)^{-s} \left(\frac{t}{\hat{\delta}_{n-l}}\right)^s. \quad \square
\end{aligned}$$

**Lemma 3.10.** *Suppose  $\rho > 1$  and  $\lambda \notin A(s)$ . Then there exists a constant  $C$  depending only on  $\rho$  and  $\lambda$  such that given  $q \in \mathbb{N}$  and  $n = l + \Gamma(q)$  for some  $l \leq \gamma_{q+1}$  and  $\hat{\delta}_n \leq t \leq \rho \cdot \delta_n$  we have,*

$$\begin{aligned}
& \#\{(\kappa, \tau) \in \mathcal{D}^n : d(\hat{\Delta}_\kappa, \hat{\Delta}_\tau) < t\} \\
& \leq 4^{\Gamma(q-1)} \cdot \left( C \cdot 2^{\gamma_q} + 4^{\gamma_q} \gamma_q \cdot \lambda^{-s} \left( \frac{\hat{\delta}_{n-l}}{\hat{\delta}_{n-l-\gamma_q}} \right)^s \right) \\
& \quad \cdot \left( C \cdot 2^l + 4^l l \cdot (\rho\lambda)^{-s} \left( \frac{t}{\hat{\delta}_{n-l}} \right)^s \right).
\end{aligned}$$

*Proof.* First note that if  $\eta \in \mathcal{D}^{n-l}$  and  $\alpha \in \mathcal{D}^l$ ,  $\hat{\Delta}_{\eta\alpha} \subseteq \hat{\Delta}_\eta$ . Hence,

$$\begin{aligned}
& \#\{(\kappa, \tau) \in \mathcal{D}^n : d(\hat{\Delta}_\kappa, \hat{\Delta}_\tau) < t\} \\
& = \sum_{(\kappa, \tau) \in \mathcal{D}^n} \chi_{\{(\kappa', \tau') \in \mathcal{D}^n : d(\hat{\Delta}_{\kappa'}, \hat{\Delta}_{\tau'}) < t\}} \\
& = \sum_{(\eta, \zeta) \in \mathcal{D}^{n-l}} \chi_{\{(\eta', \zeta') \in \mathcal{D}^{n-l} : d(\hat{\Delta}_{\eta'}, \hat{\Delta}_{\zeta'}) < t\}} \cdot \sum_{(\alpha, \beta) \in \mathcal{D}^l} \chi_{\{(\alpha', \beta') \in \mathcal{D}^l : d(\hat{\Delta}_{\eta\alpha'}, \hat{\Delta}_{\zeta\beta'}) < t\}} \\
& = \sum_{(\eta, \zeta) \in \mathcal{D}^{n-l}} \chi_{\{(\eta', \zeta') \in \mathcal{D}^{n-l} : d(\hat{\Delta}_{\eta'}, \hat{\Delta}_{\zeta'}) < t\}} \\
& \quad \cdot \#\{(\alpha, \beta) \in \mathcal{D}^l : d(\hat{\Delta}_{\eta\alpha}, \hat{\Delta}_{\zeta\beta}) < t\}.
\end{aligned}$$

By applying Lemma 3.9 along with the fact that  $t \leq \rho\delta_n \leq \rho\hat{\delta}_{n-l}$ ,

$$\begin{aligned}
& \#\{(\kappa, \tau) \in \mathcal{D}^n : d(\hat{\Delta}_\kappa, \hat{\Delta}_\tau) < t\} \\
& \leq \#\{(\eta, \zeta) \in \mathcal{D}^{n-l} : d(\hat{\Delta}_\eta, \hat{\Delta}_\tau) < t\} \cdot \left( C2^l + 4^l l \cdot (\rho\lambda)^{-s} \left( \frac{t}{\hat{\delta}_{n-l}} \right)^s \right) \\
& \leq \#\{(\eta, \zeta) \in \mathcal{D}^{n-l} : d(\hat{\Delta}_\eta, \hat{\Delta}_\tau) < \rho\hat{\delta}_{n-l}\} \\
& \quad \cdot \left( C2^l + 4^l l \cdot (\rho\lambda)^{-s} \left( \frac{t}{\hat{\delta}_{n-l}} \right)^s \right),
\end{aligned}$$

Now clearly  $\rho\hat{\delta}_{n-l} \in [\hat{\delta}_{n-l}, \rho\hat{\delta}_{n-l}]$  and so we may apply the above reasoning to the first term to obtain,

$$\begin{aligned} & \# \left\{ (\eta, \zeta) \in \mathcal{D}^{n-l} : d(\hat{\Delta}_\eta, \hat{\zeta}_\tau) < \rho\hat{\delta}_{n-l} \right\} \\ & \leq \# \left\{ (\alpha, \beta) \in \mathcal{D}^{n-l-\gamma_q} : d(\hat{\Delta}_\alpha, \hat{\beta}_\tau) < \rho\hat{\delta}_{n-l-\gamma_q} \right\} \cdot \\ & \quad \cdot \left( C \cdot 2^{\gamma_q} + 4^{\gamma_q} \gamma_q \cdot \lambda^{-s} \left( \frac{\hat{\delta}_{n-l}}{\hat{\delta}_{n-l-\gamma_q}} \right)^s \right) \\ & \leq \# \mathcal{D}^{2 \sum_{p < q} \gamma_p} \cdot \left( C \cdot 2^{\gamma_q} + 4^{\gamma_q} \gamma_q \cdot \lambda^{-s} \left( \frac{\hat{\delta}_{n-l}}{\hat{\delta}_{n-l-\gamma_q}} \right)^s \right). \end{aligned}$$

Piecing these two inequalities together completes the proof of the lemma.  $\square$

Recall that to complete the proof we must obtain the following inequality,

$$\dim_{\mathbb{C}}(\nu) = \liminf_{r \rightarrow 0} \frac{1}{\log r} \log \int \nu(B_r(x)) d\nu(x) \geq s.$$

Choose  $r \in (0, \lambda/(1-\lambda))$  and take  $n$  to be the least integer satisfying  $\hat{\delta}_n < r$ . It follows that  $r \leq \hat{\delta}_{n-1} < 2/\lambda^2 \cdot \delta_n$ . Given  $\kappa \in \mathcal{D}^n$  and a sequence  $\omega$  such that  $\kappa = \omega|_n$ , we have

$$\# \{ \tau \in \mathcal{D}^n : \hat{\Delta}_\tau \cap B_r(\pi(\omega)) \neq \emptyset \} \leq \# \{ \tau \in \mathcal{D}^n : d(\hat{\Delta}_\tau, \hat{\Delta}_\kappa) < r \}.$$

Hence,

$$\nu(B(\pi(\omega), r)) \leq \# \{ \tau \in \mathcal{D}^n : d(\hat{\Delta}_\tau, \hat{\Delta}_\kappa) < r \} \cdot 2^{-n}.$$

Since  $\nu = \mu \circ \pi^{-1}$  we have,

$$\begin{aligned} \int \nu(B_r(x)) d\nu(x) &= \int \nu(B_r(\pi(\omega))) d\mu(\omega) \\ &\leq \sum_{\kappa \in \mathcal{D}^n} \mu([\kappa]) (\# \{ \tau \in \mathcal{D}^n : d(\hat{\Delta}_\tau, \hat{\Delta}_\kappa) < r \} \cdot 2^{-n}) \\ &= 4^{-n} \# \{ (\kappa, \tau) \in (\mathcal{D}^n)^2 : d(\hat{\Delta}_\tau, \hat{\Delta}_\kappa) < r \}. \end{aligned}$$

Now note that  $\hat{\delta}_n < r \leq 2/\lambda^2 \delta_n \leq 8\delta_n$  so by Lemma 3.10 we have,

$$\begin{aligned} (5) \quad & \int \nu(B_r(x)) d\nu(x) \\ & \leq 4^{-n} \cdot 4^{\Gamma(q-1)} \cdot \left( C \cdot 2^{\gamma_q} + 4^{\gamma_q} \gamma_q \cdot \lambda^{-s} \left( \frac{\hat{\delta}_{n-l}}{\hat{\delta}_{n-l-\gamma_q}} \right)^s \right) \cdot \\ & \quad \cdot \left( C \cdot 2^l + 4^l l \lambda^{-s} \left( \frac{r}{\hat{\delta}_{n-l}} \right)^s \right) \\ & \leq \left( C \cdot 2^{-\gamma_q} + \gamma_q \cdot \lambda^{-s} \left( \frac{\hat{\delta}_{n-l}}{\hat{\delta}_{n-l-\gamma_q}} \right)^s \right) \cdot \\ & \quad \cdot \left( C \cdot 2^{-l} + l \lambda^{-s} \left( \frac{r}{\hat{\delta}_{n-l}} \right)^s \right). \end{aligned}$$

where  $q$  is chosen so that  $n = l + \Gamma(q)$  and  $0 \leq l < \gamma_{q+1}$ .

Now since  $m_q \geq \left(\frac{\log 2^{-\alpha}}{\log \lambda} - 1\right) \gamma_q$ ,

$$\frac{\hat{\delta}_{n-l}}{\hat{\delta}_{n-l-\gamma_q}} \leq \lambda^{\gamma_q+m_q} \leq 2^{-\alpha\gamma_q},$$

and provided  $l > 0$  we have

$$\frac{r}{\hat{\delta}_{n-l}} \leq \frac{8\delta_n}{\hat{\delta}_{n-l}} \leq \lambda^l.$$

Note that  $\frac{1}{2} \leq \lambda$  and since  $s \leq \frac{1}{\alpha}$ , we have  $2^{-\gamma_q} \leq (2^{-\alpha\gamma_q})^s$ . Thus, by Equation (5), if  $l > 0$  we have

$$(6) \quad \int \nu(B_r(x)) d\nu(x) \leq (2C\lambda^{-s})^2 \cdot \gamma_q (2^{-\alpha\gamma_q})^s \cdot l (\lambda^l)^s,$$

and if  $l = 0$  we have,

$$(7) \quad \int \nu(B_r(x)) d\nu(x) \leq (2C^2\lambda^{-s}) \cdot \gamma_q (2^{-\alpha\gamma_q})^s.$$

By the inequality (3) we have,

$$(8) \quad \begin{aligned} r &> \hat{\delta}_n \geq \hat{\delta}_{\Gamma(q)} \cdot \frac{\lambda}{2} \cdot \lambda^l \\ &\geq \hat{\delta}_{\Gamma(q-1)} \left(\frac{\lambda}{2}\right)^2 \cdot \lambda^{\gamma_q+m_q} \cdot \lambda^l \\ &\geq \hat{\delta}_{\Gamma(q-1)} \left(\frac{\lambda}{2}\right)^2 \cdot \lambda^{\hat{\gamma}_q+m_q} \cdot \lambda^l \\ &\geq \frac{\lambda^3}{4} \cdot \hat{\delta}_{\Gamma(q-1)} \cdot 2^{-\alpha\gamma_q-\alpha\theta_q} \cdot \lambda^l. \end{aligned}$$

Now by construction, for each  $q \in \mathbb{N}$ ,  $\gamma_q > q \log(\hat{\delta}_{\Gamma(q-1)})^{-1} \cdot \theta_q$ , so if we define

$$\iota(q) := \frac{-\log(\lambda^3/\log 4) - \log \hat{\delta}_{\Gamma(q-1)} + \theta_q \alpha \log 2}{\gamma_q \alpha \log 2},$$

we have  $\iota(q) \rightarrow 0$  as  $q \rightarrow \infty$ . Moreover, by (8),

$$\frac{\gamma_q \log 2 + l \log \lambda^{-1}}{-\log r} \geq \frac{1}{1 + \iota(q)}.$$

Substituting into Equations (6) and (7) and noting that  $q \rightarrow \infty$  as  $r \rightarrow 0$  we have,

$$\dim_{\mathbb{C}}(\nu) = \liminf_{r \rightarrow 0} \frac{1}{\log r} \log \int \nu(B_r(x)) d\nu(x) \geq s.$$

This completes the proof of the Proposition.  $\square$

#### 4. $\beta$ -SHIFTS AND A UNIFORM LOWER BOUND

Let  $1 < \beta \leq 2$ . Given a real number  $x \in \mathbb{R}$  we let  $[x]$  and  $\{x\}$  denote, respectively, the integer and fractional parts of  $x$ . Consider the  $\beta$ -transformation  $f_\beta: [0, 1) \rightarrow [0, 1)$  defined by  $x \mapsto \{\beta x\}$ . Given  $x \in [0, 1]$  we let  $\omega_n^\beta(x) := [\beta f_\beta^{n-1}(x)]$  and

$$S_\beta := \text{closure}\{(\omega_n^\beta(x))_{n \in \mathbb{N}} : x \in [0, 1)\}.$$

Let  $\pi_\beta: S_\beta \rightarrow [0, 1]$  be defined by  $(\omega_n)_{n \in \mathbb{N}} \mapsto \sum_{n \in \mathbb{N}} \omega_n \beta^{-n}$ , and let  $\sigma: S_\beta \rightarrow S_\beta$  denote the left shift operator on  $S_\beta$ . Note that  $\pi_\beta \circ \sigma = f_\beta \circ \pi_\beta$ . Parry proved in [P] that the shift space  $S_\beta$  can be written as

$$S_\beta = \{(\omega_1, \omega_2, \dots) \in \{0, 1\}^{\mathbb{N}} : \sigma^k(\omega_1, \omega_2, \dots) \leq (\omega_n^\beta(1^-))_{n \in \mathbb{N}} \ \forall k\},$$

where  $\leq$  is the lexicographical order and  $\omega_n^\beta(1^-)$  denotes the limit in the product topology of  $\omega_n^\beta(x)$  as  $x \rightarrow 1$ . Moreover, Parry proved that  $S_\beta$  is a subshift of finite type if and only if the sequence  $(\omega_n^\beta(1))_{n \in \mathbb{N}}$  terminates with infinitely many zeroes, and that a sequence  $(\omega_n)_{n \in \mathbb{N}}$  equals  $(\omega_n^\beta(1))_{n \in \mathbb{N}}$  for some  $\beta$  if and only if it satisfies

$$(9) \quad (\omega_k, \omega_{k+1}, \dots) < (\omega_1, \omega_2, \dots)$$

for all  $k > 1$ . In the set of sequences satisfying (9), the subset of sequences terminating with infinitely many zeroes is dense. This implies that the set of  $\beta$  for which the sequence  $(\omega_n^\beta(1))_{n \in \mathbb{N}}$  terminates with infinitely many zeroes is dense in  $(1, 2)$ . Hence  $S_\beta$  is a subshift of finite type for a dense set of  $\beta$ .

The following theorem allows us to transfer results from subshifts of finite type to arbitrary  $\beta$ -shifts. It is a strengthened version of Theorem 2 from [FPS], that follows immediately by replacing Lemma 6 in [FPS], by Lemma 1 in [FP2].

**Theorem 3** (Färm, Persson). *Let  $\beta \in (1, 2)$  and let  $(\beta_n)_{n \in \mathbb{N}}$  be any sequence with  $1 < \beta_n < \beta$  for all  $n$ , such that  $\beta_n \rightarrow \beta$  as  $n \rightarrow \infty$ . Suppose  $E \subset S_\beta$  and  $\pi_{\beta_n}(E \cap S_{\beta_n})$  is in the class  $\mathcal{G}^s(I)$  for all  $n$ . If  $F$  is a  $G_\delta$  with  $F \supset \pi_\beta(E \cap S_\beta)$ , then  $F$  is also in the class  $\mathcal{G}^s(I)$ .*

For  $\kappa > 0$ , we consider the sets

$$A_\beta(\kappa) = \{x \in [0, 1] : 0 \leq T_\beta^n(x) \leq \beta^{-\kappa n} \text{ infinitely often}\}.$$

We shall use the following theorem which allows us to restrict our attention to the case where  $S_\beta$  is a subshift of finite type.

**Theorem 4.** *For any  $1 < \beta \leq 2$  we have  $A_\beta(\kappa) \in \mathcal{G}^s([0, 1])$  for  $s = \frac{1}{1+\kappa}$ .*

**Remark 4.1.** *We note that the bound  $s \leq \frac{1}{1+\kappa}$  is sharp since an easy covering argument, using the fact that  $T_\beta$  has topological entropy  $\log \beta$ , shows that the Hausdorff dimension of  $A_\beta(\kappa)$  is not larger than  $\frac{1}{1+\kappa}$ .*

*Proof.* We let

$$A_{\beta,n}(\kappa) = \left\{x : 0 \leq x - y \leq 2^{-\gamma n} \text{ for some } y = \sum_{k=1}^n \frac{a_k}{\beta^k}, (a_k)_{k \in \mathbb{N}} \in S_\beta\right\},$$

and note that  $A_\beta(\kappa)$  can be written as  $A_\beta(\kappa) = \limsup_{n \rightarrow \infty} A_{\beta,n}(\kappa)$ .

By Theorem 3 it suffices to prove the theorem in the special case where  $S_\beta$  is a subshift of finite type.

When  $S_\beta$  is a subshift of finite type there are constants  $c_1$  and  $c_2$  such that

$$(10) \quad c_1 \beta^{-n} \leq |\pi_\beta([a_1, a_2, \dots, a_n])| \leq c_2 \beta^{-n}.$$

This implies that the number of cylinders of size  $n$ , denoted by  $N(n)$ , satisfies

$$(11) \quad c_2^{-1} \beta^n \leq N(n) \leq c_1^{-1} \beta^n.$$



Using these estimates we may complete the proof by following the method of [F3, Example 8.9].  $\square$

**Corollary 1.** *For any  $\lambda \in (\frac{1}{2}, 1)$  and  $\alpha > 1$  we have  $W_\lambda(\alpha) \in \mathcal{G}^s(I_\lambda)$  for  $s = \frac{-\log \lambda}{\alpha \log 2}$ .*

*Proof.* Take  $\beta = \lambda^{-1}$  and  $\kappa = \frac{\alpha \log 2}{\log \beta} - 1$ . It follows that  $A_\beta(\kappa) \subset W_\lambda(\alpha)$ , so  $W_\lambda(\alpha) \in \mathcal{G}^s([0, 1])$  follows immediately from Theorem 4. Now, the self-similar structure of  $W_\lambda(\alpha)$  implies that  $W_\lambda(\alpha) \in \mathcal{G}^s(I_\lambda)$ .  $\square$

## 5. COVERING ARGUMENTS AND UPPER BOUNDS

Each of the upper bounds from Theorem 1 parts (1), (3) and (5) will rely on the following simple relationship between the growth in the number of  $n$ th level  $\lambda$  sums and the dimension of  $W_\lambda(\alpha)$ . Given  $\lambda \in (\frac{1}{2}, 1)$  and  $n \in \mathbb{N}$  we let

$$F_{\lambda,n} := \left\{ \sum_{k=1}^n a_k \lambda^k : a_k \in \{0, 1\} \right\},$$

and let

$$\tau(\lambda) := \limsup_{n \rightarrow \infty} \frac{\log \#F_{\lambda,n}}{n \log 2}.$$

**Lemma 5.1.** *For all  $\lambda \in (\frac{1}{2}, 1)$  and  $\alpha > 1$  the Hausdorff dimension of  $W_\lambda(\alpha)$  is bounded above by  $\tau(\lambda)/\alpha$ .*

*Proof.* This may be deduced by a standard covering argument. See for example the first paragraph in the proof of Jarník's theorem from [F3, Section 10.3].  $\square$

Our first corollary establishes Theorem 1 (1).

**Corollary 2.** *For all  $\lambda \in (\frac{1}{2}, 1)$  and  $\alpha > 1$  the Hausdorff dimension of  $W_\lambda(\alpha)$  is bounded above by  $1/\alpha$ .*

*Proof.* This is immediate from Lemma 5.1 combined with the fact that  $\#F_{\lambda,n} \leq 2^n$  so  $\tau(\lambda) \leq 1$  for all  $\lambda \in (\frac{1}{2}, 1)$ .  $\square$

Our second corollary establishes Theorem 1 (3).

**Corollary 3.** *There exists a dense family  $\Gamma \subset (\frac{1}{2}, 1)$  such that for all  $\lambda \in \Gamma$ ,  $\dim W_\lambda(\alpha) < 1/\alpha$ .*

*Proof.* Our approach is based on [SS]. We let  $\Gamma$  denote the set of  $\lambda \in (\frac{1}{2}, 1)$  such that for some finite word  $(\omega_i)_{i=1}^n \in \{0, 1\}^n$  we have  $1 = \sum_{i=1}^n \omega_i \lambda^i$ . To see that  $\Gamma$  is dense in  $(\frac{1}{2}, 1)$  first fix  $\lambda_0 \in (\frac{1}{2}, 1)$  and  $\epsilon \in (0, 1 - \lambda_0)$ . Then there exists an infinite string  $(\omega_i)_{i=1}^\infty \in \{0, 1\}^\mathbb{N}$  with  $1 = \sum_{i=1}^\infty \omega_i \lambda_0^i$ . Let  $k$  be the smallest  $q$  with  $\omega_q = 1$  and choose  $n$  so that  $\sum_{i=1}^n \omega_i \lambda_0^i > 1 - \epsilon^k$ . Then for some  $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$  we have  $\sum_{i=1}^n \omega_i \lambda^i = 1$ , so  $(\lambda_0, \lambda_0 + \epsilon) \cap \Gamma \neq \emptyset$ .

By Lemma 5.1 it suffices to show  $\tau(\lambda) < 1$  for all  $\lambda \in \Gamma$ . But if  $\lambda \in \Gamma$  then for some finite word  $(\omega_i)_{i=1}^n \in \{0, 1\}^n$  we have  $\lambda^{q(n+1)+1} =$

$\sum_{i=1}^n \omega_i \lambda^{i+1+q(n+1)}$  for all  $q \in \mathbb{N}$ . It follows that for all  $q \in \mathbb{N}$ ,

$$\begin{aligned} F_{\lambda, q(n+1)} &= \\ &= \left\{ \sum_{i=0}^{q-1} \sum_{j=1}^{n+1} a_{i(n+1)+j} \lambda^{i(n+1)+j} : \right. \\ &\quad \left. (a_{i(n+1)+1}, \dots, a_{i(n+1)+(n+1)}) \in \{0, 1\}^{n+1} \right\} \\ &= \left\{ \sum_{i=0}^{q-1} \sum_{j=1}^{n+1} a_{i(n+1)+j} \lambda^{i(n+1)+j} : \right. \\ &\quad \left. (a_{i(n+1)+1}, \dots, a_{i(n+1)+(n+1)}) \in \{0, 1\}^{n+1} \setminus \{(1, 0, \dots, 0)\} \right\}. \end{aligned}$$

Thus, for each  $q$  we have

$$\#F_{q(n+1)} \leq (2^{n+1} - 1)^q,$$

so for all  $l \in \mathbb{N}$ ,

$$\#F_{l, \lambda} \leq \#F_{[l/(n+1)](n+1)} \leq (2^{n+1} - 1)^{[l/(n+1)]}.$$

Thus,  $\tau(\lambda) \leq \log(2^{n+1} - 1) / (n+1) \log 2 < 1$ .  $\square$

Finally we complete the proof of Theorem 1 (5).

**Definition 5.1.** A *multinacci number* is a positive real  $\lambda$  which satisfies an equation of the form  $\lambda^m + \dots + \lambda = 1$  for some  $m \in \mathbb{N}$ .

We note that there are countably many multinacci numbers, all of which are contained within the interval  $(\frac{1}{2}, 1)$ . The largest multinacci number is the golden ratio  $\frac{\sqrt{5}-1}{2}$ .

**Theorem 5.** Let  $\lambda$  be a multinacci number. Then the Hausdorff dimension of  $W_\lambda(\alpha)$  is  $-\frac{\log \lambda}{\log 2} \frac{1}{\alpha}$ .

*Proof.* Put

$$\begin{aligned} S_1 &: x \mapsto \lambda x, \\ S_2 &: x \mapsto \lambda(x+1). \end{aligned}$$

Let us first consider the case  $m = 2$ . Then  $\lambda = \frac{\sqrt{5}-1}{2}$  and  $S_1 \circ S_2 \circ S_2 = S_2 \circ S_1 \circ S_1$ . Hence, when defining  $W_\lambda(\alpha)$  we need only consider sequences where the word 011 is forbidden, since replacing the word 011 in a sequence by the word 100, yields the same point. Hence, if we put

$$F_{\lambda, n} = \left\{ \sum_{k=1}^n a_k \lambda^k : a_k \in \{0, 1\} \right\},$$

then we have

$$F_{\lambda, n} = \left\{ \sum_{k=1}^n a_k \lambda^k : a_k \in \{0, 1\}, (a_k, a_{k+1}, a_{k+2}) \neq (0, 1, 1) \right\}.$$

The subshift in which 011 is forbidden is a subshift of finite type, with adjacency matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

One checks that  $\lambda^{-1} = \frac{\sqrt{5}+1}{2}$  is the largest eigenvalue of  $A$ . Hence there is a constant  $K$  such that  $\#F_{\lambda,n} < K\lambda^{-n}$ . Hence  $\tau(\lambda) = -\frac{\log \lambda}{\log 2}$ , so by Lemma 5.1 the Hausdorff dimension of  $W_\lambda(\alpha)$  is at most  $-\frac{\log \lambda}{\log 2} \frac{1}{\alpha}$ . But by Corollary 1 it is at least  $-\frac{\log \lambda}{\log 2} \frac{1}{\alpha}$ .

For a general  $m \geq 2$  we proceed similarly. Assume  $\lambda$  is such that  $S_1 \circ S_2^m = S_2 \circ S_1^m$ . This implies that  $\lambda$  satisfies the equation

$$(12) \quad \lambda^m + \lambda^{m-1} + \dots + \lambda = 1.$$

As before, for the set  $F_{\lambda,n}$ , we need only consider sequences where the word

$$0 \underbrace{11 \dots 1}_m$$

is forbidden. This is again a subshift of finite type, and it can be represented using a  $2^m \times 2^m$  adjacency matrix given by

$$A = \begin{bmatrix} 1 & 1 & & & & & \\ & & 1 & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & 0 \\ 1 & 1 & & & & & \\ & & 1 & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & 1 \end{bmatrix}.$$

By the Perron–Frobenius theorem, the eigenvalue of largest modulus of this matrix, is a positive number, and it has a corresponding eigenvalue with positive elements. Let  $v = [v_1 \dots v_{2^m}]^T$  be such an eigenvector and let  $\mu$  be the eigenvalue. It is not hard to see that the equation  $Av = \mu v$  implies that

$$\begin{aligned} v_1 &= v_{2^{m-1}+1}, \\ v_2 &= v_{2^{m-1}+2}, \\ &\vdots \\ v_{2^{m-1}-1} &= v_{2^m-1}. \end{aligned}$$

Let  $1 \leq k < 2^{m-2} - 1$ . Looking at row  $k$  and row  $2^{m-2} + k$  in the equation  $Av = \mu v$ , we see that  $v_k = v_{2^{m-2}+k}$ . Continuing in this fashion we end up in the conclusion that all  $v_k$  for odd  $k$  are equal. Without loss of generality we can therefore assume that  $v_k = 1$  for odd  $k$ .

If we look at the first row of the matrixes in the equation  $Av = \mu v$ , we see that  $\mu v_1 = v_1 + v_2 = 1 + v_2$ . We continue, and looking at the second row, we see that  $\mu v_2 = v_3 + v_4 = 1 + v_4$ . Hence we have

$$\mu = \mu v_1 = 1 + v_2 = 1 + \mu^{-1}(1 + v_4).$$

Similarly we get  $\mu v_4 = 1 + v_8$ , and so

$$\mu = 1 + \mu^{-1} + \mu^{-2}(1 + v_8).$$

We can continue this process, using the equations

$$\mu v_{2^k} = 1 + v_{2^{k+1}},$$

that are valid for  $0 \leq k \leq m-3$ , to conclude

$$\mu = 1 + \mu^{-1} + \cdots + \mu^{-m+2}(1 + v_{2^{m-2}}).$$

But we have  $\mu v_{2^{m-2}} = v_{2^{m-1}-1} = 1$ , hence

$$\mu = 1 + \mu^{-1} + \cdots + \mu^{-m+2} + \mu^{-m+1},$$

or equivalently

$$\mu^m = 1 + \mu + \cdots + \mu^{m-1}.$$

Comparing with the equation (12), this implies that we have  $\mu = \lambda^{-1}$ .

The rest is just as for the case  $m = 2$  above. We have that  $\#F_{\lambda,n} < K\mu^n = K\lambda^{-n}$ , and therefore  $\tau(\lambda) = -\frac{\log \lambda}{\log 2}$ , so by Lemma 5.1 the Hausdorff dimension of  $W_\lambda(\alpha)$  is at most  $-\frac{\log \lambda}{\log 2} \frac{1}{\alpha}$ .  $\square$

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